DERIVATION OF THE FREQUENCY FUNCTION OF STELLAR FLARES IN A STAR CLUSTER*

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Data on flare stars in the Pleiades indicate directly that the mean frequency of flares differs strongly for different flare stars. The solution is found to the problem of determining the distribution function of the mean frequencies of flare stars on the basis of statistical data on all flares observed in a given aggregate. The need to do this arises because it is in practice impossible to determine the mean flare frequency for individual stars. The problem is solved by using the chronology of discoveries ("first flares") and the chronology of confirmations (observations of "second flares"). A concrete solution is found for the flare stars in the Pleiades. The results obtained are far from being definitive. However, the possibility of using the chronology of discoveries of flare stars and the chronology of confirmations to obtain serious information about a cluster appears sufficiently striking to the author for him to have wished to dedicate the present paper to the eminent astrophysicist and physicist Professor H. Alfvén on the occasion of his 70th birthday.

1. In astronomy, one frequently encounters situations in which certain quantities can be measured directly and one wishes, from these directly measured quantities, to deduce the values of other quantities or functions of more fundamental significance that are related to the first but cannot be directly measured. Such cases frequently lead to the solution of mathematical problems known under the general heading of "inverse problems of mathematical physics". Sometimes the problem can be reduced to the solution of a mathematical problem that does not require a particularly complicated formalism.

As an example from classical astronomy, we can mention the problem solved by Gauss of determining the elements of a planetary orbit from three observations; this problem is the inverse of the "direct problem" of calculating the ephemeris of a planet from the given elements of its orbit.

A simple example from the field of stellar astronomy is the problem of determining the spatial star density in a globular cluster as a function of the distance to the center of the cluster from observations of the star density distribution as projected onto the celestial sphere. It is well known that this problem reduces to the solution of an Abel integral equation. Of course, certain assumptions are also made in the solution of this problem, it being assumed, in particular, that deviations from a spherical distribution can be ignored.

The problem of a globular cluster brings out rather well the statistical nature of the problem and the difficulties related to this. The point is that the surface star density cannot be found from the observations with arbitrarily great accuracy, this last being limited by random fluctuations. The resulting uncertainty (inaccuracy) of the given function gives rise to an even greater uncertainty in the required function (the spatial density).

A third example, in which the difficulties associated with the statistical nature of the function obtained from observations are even more obvious, is the problem of finding the distribution function $\varphi(\xi, \eta, \zeta)$ of the spatial velocities of stars from observations of the distribution function $\psi(\nu, \alpha, \delta)$ of the radial velocities of stars in different parts of the celestial sphere. This problem was posed by Eddington and solved by the present author about 40 years ago [1].

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A fourth example is the problem of determining the total number of flare stars in a cluster (or in a stellar association) when one knows the numbers of stars that in a definite time interval τ have been observed to have one, two, three, etc., flares each but there still remains an unknown and large number of undiscovered flare stars. This inverse problem admits a simple mathematical formulation in the case when the flares of each of the flare stars form a homogeneous (with constant mean frequency) Poisson sequence. In this case, when the mean flare frequency is the same for all the flare stars, it is sufficient to know only the numbers m_1 and m_2 of stars that are observed to have one and two flares each, respectively. Then the number m_0 of stars not yet observed to have flares is given by the simple expression

$$m_0 = \frac{m_1^2}{2m_2}. \tag{1}$$

Actually, the relation (1) holds only between the mathematical expectations of the numbers m_0, m_1, and m_2. However, for want of better, when calculating the mathematical expectation of m_0 we usually substitute in (1), not the mathematical expectations of m_1 and m_2, but their observed values during the time τ.

2. However, examination of the problem reveals that in at least some star aggregates there are stars with *strongly differing mean flare frequencies* [3]. The Pleiades is such an aggregate. As a result, the use of Eq. (1) gives a rough answer suitable only for a first approximation, and to describe the aggregate of flare stars one needs to know not only the total number N of flare stars but also their distribution function $f(\nu)$ with respect to the frequency ν. Even more complete information would be provided by these numbers for each interval of apparent magnitude (at the light minimum). To solve this problem exactly, one should use the direct method, i.e., one ought to observe the clusters for so long that each of the flare stars has such a large number of flares that it is possible to estimate the mean frequency for each individual star. For the stellar aggregates containing flare stars (for example, Pleiades, Orion) currently under observation by the astronomers this is impossible, since the material collected during several years of observations still leaves undetected an appreciable number of flare stars, and the stars that have been observed to have more than two flares constitute only a very small fraction of all the flare stars. Therefore, we pose the problem of statistical determination of the *total number* of flare stars and their distribution with respect to the flare frequencies without prior determination of the mean frequencies for each star. For the time being, we assume that this distribution does not depend on the star's magnitude at the light minimum. In principle, the method of solution will also be suitable for finding the same data for individual magnitude intervals.

Suppose that at time $t = 0$ we begin to observe the flares in the aggregate, and let $P(t)$ be the probability that in the interval $(0, t)$ there is at least one flare of a flare star chosen randomly in the aggregate. For the time being, we assume that the observations are made continuously. This assumption is made only to simplify the arguments. We shall see that this condition can be relaxed and the results remain valid provided the method of measuring the time from the start $t = 0$ is modified.

Then $P(t)$ can be expressed in terms of the distribution function $f(\nu)$ of the Poisson parameter ν (the mean frequency of flares of the star) in the form

$$P(t) = 1 - \int_0^\infty e^{-\nu t} f(\nu)\, d\nu. \tag{2}$$

Obviously, $NP(t)$ is the mathematical expectation of the number of "first" flares that take place up to time t, i.e., it is the mathematical expectation of the number of flare stars discovered up to time t (it is assumed that no flare star had been discovered before $t = 0$).

Obviously, the derivative

$$n_1(t) = N\frac{dP(t)}{dt} \tag{3}$$

is the mathematical expectation of the number of stars that have their "first" flares (which are observed) in unit time at time t. It follows from (2) and (3) that

$$n_1(t) = N \int_0^\infty e^{-\nu t} \nu f(\nu)\, d\nu, \tag{4}$$

from which we conclude

$$\frac{n_1(t)}{n_1(0)} = \frac{\int_0^\infty e^{-\nu t} \nu f(\nu)\, d\nu}{\int_0^\infty \nu f(\nu)\, d\nu}.$$

Introducing the mean value $\bar{\nu}$ of the mean flare frequency,

$$\bar{\nu} = \int_0^\infty \nu f(\nu)\, d\nu, \tag{5}$$

we obtain

$$\frac{n_1(t)}{n_1(0)} = \frac{1}{\bar{\nu}} \int_0^\infty e^{-\nu t} \nu f(\nu)\, d\nu. \tag{6}$$

The left-hand side of this equation can be found from observations. Since we do not know $\bar{\nu}$ in this equation, we can, inverting the Laplace transformation, find $f(\nu)$ only to within a constant factor. Since, however, $f(\nu)$ is a probability density, the normalization

$$\int_0^\infty f(\nu)\, d\nu = 1 \tag{7}$$

always enables us to determine this constant factor.

Since every Poisson process consists of mutually independent events, we can cut out finite intervals of the time axis and join up the remaining intervals and still have a Poisson sequence of flares for each star provided the cutting out is made completely independently of the existing concrete realization of the process. As a result, interruptions in the continuous observation made necessary by the conditions of observation (daylight, bad weather, etc.) are unimportant. It is only necessary to take as the time elapsed since the start $t = 0$ of observations the time t of the observations (the sum of the exposures) made since $t = 0$.

We now draw attention to the physical meaning of the left-hand side of Eq. (6). For t near zero, all the observed flares are first flares of the corresponding stars in the period of observation, and therefore the number $n_1(0)$ is simultaneously both the number of all flares that occur in the aggregate and the number of all "first" flares in unit time. On the other hand, the rate *of all flares* is stationary by the hypothesis that it is a sum of Poisson processes. Therefore, $n_1(0)$ is also the number of *all flares* in unit time at any time. Therefore, the left-hand side of Eq. (6) is the *relative fraction of "first" flares $n_1(t)$ among all flares that occur in unit time.* If $t = 0$ is the time when observation of the aggregate begins, then $n_1(t)$ is the number of flare stars newly discovered in unit time. Therefore, Eq. (6) also has the following meaning: the relative fraction of newly discovered flare stars among all stars that have flares in unit time is equal, to within a constant factor, to the Laplace transform of the function $\nu f(\nu)$.

3. Suppose that to solve Eq. (6) we determine from the observational data the ratio $n_1(t)/n_1(0)$ by counting the first flares for six time intervals, i.e., we split the complete time interval of the observations into six equal parts. Since the corresponding numbers $n_1(t)$ for the Pleiades have by now reached about 80 on average, the random deviations of the observed values $n_1(t)$ from their mathematical expectations must be about 10% of the actual value. Under such conditions, when the given function is given at only six points and, moreover, with such low accuracy, the inversion of the Laplace transformation leads to *very large* relative errors in the determination of the distribution function $f(\nu)$.

One can however hope to improve the situation considerably by using the possibility of indirect determination of the function $n_1(t)$ from other observational data.

For this, we write down a formula for the expected number N_2 of stars for which both the first and the second flare are observed during the time t, i.e., during this time two or more flares are observed:

$$N_2 = N\int_0^\infty f(\nu)(1 - e^{-\nu t} - e^{-\nu t}\nu t)\, d\nu \tag{8}$$

or

$$N_2 = N_1 - N\int_0^\infty f(\nu)\, e^{-\nu t}\, \nu t d\nu, \tag{9}$$

where

$$N_1 = NP(t) = N\int_0^\infty f(\nu)(1 - e^{-\nu t})\, d\nu. \tag{10}$$

It is readily seen from (9) that

$$N_2 = N_1 + Nt\frac{d}{dt}\int_0^\infty e^{-\nu t} f(\nu)\, d\nu. \tag{11}$$

And since on the basis of (10) and (7)

$$N\int_0^\infty f(\nu)\, e^{-\nu t} d\nu = N - N_1(t),$$

we obtain

$$N_2 = N_1 + t\frac{d}{dt}(N - N_1). \tag{12}$$

Since $dN/dt = 0$,

$$N_2 = N_1 - t\frac{dN_1}{dt}. \tag{13}$$

Regarding (13) as a differential equation for N_1, we obtain its solution $N_1(t)$ as a function of the time:

$$N_1(t) = Ct - t\int_0^t \frac{N_2(u)du}{u^2}. \tag{14}$$

To determine the value of the constant C, we differentiate this equation with respect to t and remember that the derivative of the left-hand side is

$$n_1(t) = C - \int_0^t \frac{N_2(u)\,du}{u^2} - \frac{N_2(t)}{t}.$$

At t = 0,

$$n_1(0) = C,$$

since $N_2(t)$ must be of order t^2 at small t. Thus,

$$N_1(t) = n_1(0)\,t - t\int_0^t \frac{N_2(u)\,du}{u^2}. \tag{15}$$

Differentiating with respect to t and then integrating by parts, we obtain

$$n_1(t) = n_1(0) - \int_0^t \frac{dN_2(u)}{u}.$$

Since $dN_2(t) = n_2(t)dt$, where $n_2(t)$ is the number of second flares observed in unit time, we can obviously rewrite this equation in the form

$$n_1(t) = n_1(0) - \int_0^t \frac{n_2(u)\,du}{u}. \tag{16}$$

Thus, we obtain the values of the function $n_1(t)$ in terms of the statistics of the times of second flares, the data on which is obviously to some extent independent of the distribution of the first flares. In addition, the values of $n_1(t)$ are determined through (16) by integrating the observed function $n_2(t)$, which leads to smaller relative fluctuations in the values obtained for $n_1(t)$. We can therefore expect this second method of empirical determination of $n_1(t)$ to help in the more exact calculation of the left-hand side of Eq. (6). This, in its turn, is extremely important for a more reliable determination of the solution of this equation.

4. On the basis of what we have said, we can propose the following program for determining the distribution function $f(\nu)$ of the mean flare frequencies.

a) All the flares are arranged in chronological order and the "first flares" are separated; by direct counting we then find the function $n_1(t)/n_1(0)$, which is the fraction of first flares among all flares as a function of the time. This time is not calendar time but the time measured from the start of observations by a clock that runs only during observations of the aggregate. For this, the complete duration of the observations is divided into intervals which are sufficiently small that in them the mathematical expectation of the change in $n_1(t)$ is small compared with $n_1(t)$ itself. For example, each interval may be of order 30 or 50 h. In these intervals, the $n_1(t)$ counts are made.

b) The function $n_2(t)$ is found in the same way and $n_1(t)$ is found by numerical integration on the basis of Eq. (16).

Because of the instability of the problem of inverting Eq. (16) — this is what mathematicians call an improperly posed problem — it is desirable to use both methods of determining $n_1(t)/n_1(0)$ in order to obtain confirmation and also to be able to average.

c) Inverting the Laplace transformation on the basis of Eq. (6), we find $f(\nu)$ from $n_1(t)/n_1(0)$. The normalization (7) must be used.

d) The correctness of the solution found for $f(\nu)$ can then be tested on the basis of other observational data obtained independently of $n_1(t)$ and $n_2(t)$. For example, using the expression for the mathematical expectation m_k of the numbers of stars that are observed to have k flares during the complete time τ,

$$m_k = N \int_0^\infty e^{-\nu\tau} f(\nu) \frac{(\nu\tau)^k}{k!}\, d\nu, \tag{17}$$

we can, knowing $f(\nu)$, determine the ratios m_k/m_1 and then make a comparison with the observed realization of the numbers m_k.

5. This method was applied to the flare stars observed in the Pleiades. We decided to ignore all flares for which the photographically observed amplitude is less than one magnitude. Thus, we are finding the distribution function of such "large" flares. We did this in order to eliminate completely spurious flares due to random local variations of the sensitivity on the photographic plates.

According to the data given to me by É. S. Parsamyan and based on the catalog of all flares taken from the literature, the total number of such "large" flares is 822. The corresponding total observation time is 2625 h. Unfortunately, we do not know in all cases how these hours of observation are placed in time since the authors usually *do not publish the times of all photographs* but only publish the *times of observed* flares and the total duration of all exposures during the relevant period. A difficulty therefore arose in determining the times of the flares in our nominal time t. We are however justified in assuming that the flare activity was uniform during the complete time τ of the observations and moreover the published times make it possible to arrange them in chronological sequence; it is therefore sufficient to assume that the time of the flare in the nominal time is proportional to the number z of the flare, i.e.

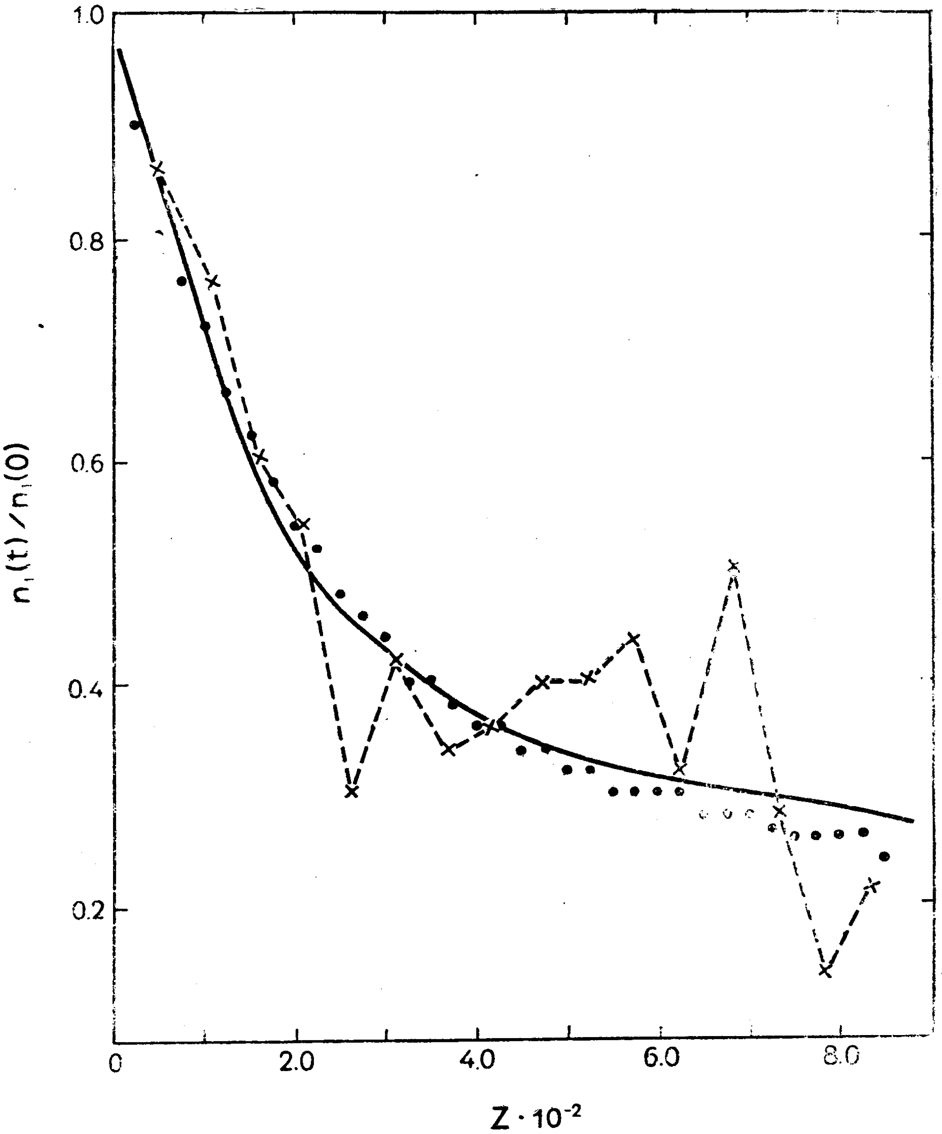

Fig. 1. Dependence of $n_1(t)/n_1(0)$ on z. The crosses are the values of $n_1(t)/n_1(0)$ obtained directly from observations. The smooth curve is an interpolation by hand based on the same values (see the text). The points are the values calculated in accordance with Eq. (16) on the basis of the counts of "second" flares.

$$t = z \frac{\tau}{z(\tau)}. \tag{18}$$

In Fig. 1, the dashed curve represents data on $n_1(t)/n_1(0)$ obtained from direct counts, while the smooth curve is an interpolation based on the same data but under the assumption that the mathematical expectation of the fraction of first flares among all flares must decrease monotonically with the time.

The appreciable fluctuations of the dashed curve with respect to the monotonically decreasing curve illustrates well our earlier assertion that it is difficult to determine the mathematical expectations of $n_1(t)/n_1(0)$ on the basis of only direct counts.

Further, on the basis of direct counts, we determined the function $n_2(t)$. Although the function $n_2(t)$ obtained from the observations also contains large fluctuations, these are strongly smoothed by the integration in accordance with Eq. (16). We were suprised how well the values $n_1(t)/n_1(0)$ obtained in this way and plotted in the figure in the form of points coincided with the previously drawn interpolation curve.

All these data can be satisfactorily described by the simple interpolation formula

$$\frac{n_1(t)}{n_1(0)} = \frac{1}{(1+0.00260\,t)^{2/3}}. \tag{19}$$

It is readily seen that the solution of Eq. (6) with this analytic form of the left-hand side must have the form

$$f(\nu) = Ce^{-\nu s}\,\nu^{-4/3}, \tag{20}$$

where the parameter s, which has the dimensions of time, is equal to

$$s = 385\ \mathrm{h}. \tag{21}$$

We should make two remarks concerning the obtained distribution function $f(\nu)$: a) an appreciable fraction of the flare stars has mean frequencies lower than 0.001 h^{-1}; b) there is a singularity at the point $\nu = 0$, as a result of which the integral over the complete frequency interval diverges. Of course, at small ν the true function must behave differently. It is obvious that observations which have lasted for only 2625 h cannot give any reliable information about the statistics of flares for stars for which the mean interval between flares is greater than, say, 2500 h. For such frequencies, the expression (20) is a purely formal result. It would therefore be equally incorrect to assume that there are a large or a small number of such stars with low flare frequency. We recognize here that the mathematical problem is improperly posed and we feel the inadequacy of the corresponding observational data.

Particularizing, we can say that the true function must have the form

$$f(\nu) = Ce^{-\nu s}\,\nu^{-4/3}\,g(\nu), \tag{22}$$

where $g(\nu)$ can be taken equal to unity for large ν (say for $\nu > 0.001\ h^{-1}$) and tends rapidly to zero as $\nu \to 0$. However, we cannot yet describe quantitatively the behavior of $g(\nu)$ at small ν.

This last circumstance prevents our determining the value of C on the basis of the normalization (7). Nevertheless, the product NC, which occurs in the expression

$$NCe^{-\nu s}\,\nu^{-4/3}\,g(\nu)\,d\nu \tag{23}$$

for the mathematical expectation of the number of stars in the frequency interval, can be estimated as follows.

From (4) for t = 0, we find

$$n_1(0) = N\int_0^\infty f(\nu)\,\nu\,d\nu = NC\int_0^\infty e^{-\nu s}\,\nu^{-1/3}g(\nu)\,d\nu. \tag{24}$$

Since the integral on the right-hand side of (24) also converges for $g(\nu) \equiv 1$, by assuming $g(\nu) = 1$ also in the part corresponding to small ν we do not introduce any large error. Integrating, we obtain under this assumption

$$NC = n_1(0)\,\frac{s^{2/3}}{\Gamma(2/3)}.$$

Substituting here s = 385 h, $n_1 = 0.313\ h^{-1}$, and $\Gamma(2/3) = 1.354$, we obtain NC = 12.2. Therefore, (23) for high frequencies has the form

$$dN = 12.2\,e^{-\nu s}\,\nu^{-4/3}\,d\nu,$$

where ν must be expressed in units of h^{-1}.

This then is the expression for the absolute number of flare stars in the Pleiades for different frequency intervals.

TABLE 1. Calculated Numbers $N(\nu_0)$ of Stars in the Pleiades with Mean Flare Frequencies Greater than ν_0

$s\nu_0$	Π	$N(\nu_0)$
5.0	77	0.04
4.0	96	0.18
3.0	128	0.7
2.0	192	3.1
1.0	385	17
0.30	1280	94
0.10	3850	245
0.05	7700	385
0.02	19250	634
0.01	38500	886

TABLE 2

k	$(n_k)_{obs}$	m_k
1	213	(213)
2	62	62
3	46	30
4	20	17
5	9	11
6	29	30

For the total number of stars having frequency greater than some ν_0, we obtain

$$N(\nu_0) = 12.2 \int_{\nu_0}^{\infty} e^{-\nu s} \nu^{-4/3} d\nu$$

or, replacing ν by the variable $x = \nu s$,

$$N(\nu_0) = 12.2 s^{1/3} \int_{\nu_0 s}^{\infty} e^{-x} x^{-4/3} dx,$$

and using the value of s given by (21), we obtain

$$N(\nu_0) = 88.7 \int_{385\nu_0}^{\infty} e^{-x} x^{-4/3} dx. \qquad (25)$$

The integral on the right-hand side can be found numerically for different values of the parameter $\nu_0 s$. We then obtain Table 1 of the $N(\nu_0)$ values, in which Π is the mean interval between flares at the frequency ν_0 expressed in hours.

Of course, the last row of Table 1 is a crude extrapolation. Nevertheless, it is worth drawing attention to the two following unexpected circumstances.

a) The majority of flare stars have mean interval between flares longer than 5 000 h.

b) Although the numerical value of $N(\nu_0)$ in the last row of Table 1 is extremely unreliable, we must apparently assume that at least some stars have mean intervals exceeding 20 000 h.

Both of these conclusions refer to intervals between flares whose amplitude exceeds one magnitude.

6. The obtained distribution function (20) enables us, as we said above, to use Eq. (17) to calculate the ratios m_2/m_1, m_3/m_1, ... of the mathematical expectations and compare them with the observations, which we do in Table 2. Since the value of the constant C in Eq. (20) remains unknown because the required function is uncertain at low frequencies, the (calculated) numbers m_k given in the third column of Table 2 were obtained by multiplying the ratios m_k/m_1 deduced from Eqs. (17) and (20) by the observed value m_1 ($k = 2, 3, \ldots$). It is for this reason that the value of m_1 in the third column is equal to the value of the same quantity in the second column and put in brackets.

7. In the investigations of flare stars made by the Byurakan group we have always wanted to determine the total number of all flare stars, and this requires knowledge of

the number n_0 of flare stars that do not have any flares during the time of observations. In the approximation in which it is assumed that all the flare stars have the same mean frequency, the value of n_0 was obtained from Eq. (1). But we have seen that the new values are distributed in a fairly wide interval, and the infinite increase of $f(\nu)$ in accordance with Eq. (20) at small ν mentioned above leads to a divergence of the integral (17) at $k = 0$. Evidently, we must change the formulation of the problem and attempt to determine, not the number of all flare stars, but only the number for which the frequencies exceed a certain ν_0.

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INFLUENCE OF ELECTRON SCATTERING ON THE CONTINUOUS SPECTRUM OF A STAR

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The photospheres of hot stars (or other astrophysical objects) in which scattering of the radiation by free electrons is important are considered. The ratio of the coefficient of electron scattering to the absorption coefficient is assumed to be an arbitrary function of the optical depth. A linear integral equation is obtained that directly determines the intensity of the radiation leaving the star. This equation is solved numerically for the case of an isothermal photosphere in which the density varies in accordance with the barometric law. In this case, an asymptotic expression is also given for the radiation intensity.

It is well known that the scattering of radiation on free electrons plays an important role in the photospheres of hot stars. Such scattering is particularly important in the photospheres of Wolf—Rayet stars. It was first investigated by Ambartsumyan [1].

Estimates show (see, for example, [2] and [3]) that electron scattering must also be taken into account when one is considering radiative transfer in the envelopes of supernovae.

In a number of investigations ([4, 5], for example) the electron scattering mechanism has also been invoked to explain the energy distribution in the spectra of x-ray sources.

Thus, the problem of determining the influence of electron scattering on the continuum is encountered in the study of various astrophysical objects. Initially [6, 7], this problem was considered under the assumption that the ratio of the coefficient of electron scattering to the absorption coefficient does not change in the medium. Recently [3] radiative transfer in a homogeneous sphere was investigated under this assumption. However, the ratio in question usually depends strongly on the depth, and in [8, 9] the problem was solved for some special cases of this dependence.

In the present paper, as in [9], we consider radiative transfer in a stellar photosphere in which light is not only emitted and absorbed but is also scattered by free electrons. As a result, we find the energy distribution in the continuum of the star. However, in contrast to [9], the ratio of the electron scattering coefficient to the absorption coefficient is taken in a different form, and the problem is solved in a different way.

Although we are concerned with stellar photospheres in this paper, its results can also be applied to other astrophysical objects.

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