ON THE DERIVATION OF THE FREQUENCY FUNCTION OF SPACE VELOCITIES OF THE STARS FROM THE OBSERVED RADIAL VELOCITIES.

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(Communicated by Sir Arthur Eddington)

One of the most important problems of stellar statistics is the derivation of the frequency function of the space velocities of stars of various spectral types and of different absolute magnitudes in our neighbourhood. The direct solution of this problem requires the knowledge of the space velocities of a great number of stars. The derivation of the space velocity of a given star is possible only in the case when three different quantities are measured : the radial velocity, the proper motion and the parallax. These quantities are measurable with different relative degrees of accuracy and are exposed to systematic errors of quite different kinds. For some important groups of stars (for example B-type stars) we have very few reliable individual parallaxes. In general, the number of reliable parallaxes is generally small, and comparatively few stars with known radial velocities have known parallaxes.

Therefore several writers have made the attempts to obtain some knowledge about the distribution law of space velocities from the radial velocities alone. However, in every case some more or less arbitrary form of this law was assumed, and the problem was restricted to the finding of numerical values of some constant parameters entering in this form of distribution law. In the majority of cases these constants are the elements of the velocity ellipsoids.

Owing to the relative uniformity of the catalogues of the radial velocities, the results of the statistical investigations based on them are almost free from the influence of systematic errors.

It seems desirable, then, to try to solve the problem of derivation of the frequency function of space velocities from the distribution of radial velocities without making any hypothesis about the form of this function.

So far as it is known to the writer, this problem not only remains yet unsolved, but is not even discussed in any detail. The purpose of the present paper is to derive the general formula which enables us to compute the frequency function of space velocities from the distribution of radial velocities.

It will be shown that the frequency function of the space velocities is the solution of an integral equation. In this equation the observed frequency function of the radial velocities for the different parts of the sky enters as the known function. We give below the derivation of the equation and its solution.

The Fundamental Assumption.—We shall assume that the different elementary volumes of space in our neighbourhood have practically identical frequency functions of the space velocities. In actual cases, when relatively rare types of stars (for example Cepheids) are considered, it is necessary to consider also the distant stars, since the number of stars of such types in our neighbourhood is very small. In such cases some corrections for the difference between the frequency functions in various parts of the galaxy are required. The actual process of introducing these corrections is beyond the scope of the present paper. We suppose that the radial velocities of a sufficiently large number of near stars situated in different parts of the sky are given, and our aim is to derive the frequency function of the space velocities from these radial velocities.

We shall consider first the two-dimensional problem. It is of special interest, since some types of stars are strongly concentrated to the galactic plane and the z-components of their velocities are small.

The Two-dimensional Problem.—If the stars are distributed over a plane and we are situated in the same plane, we may measure for each star the radial velocity V and its apparent position or azimuth, reckoned from some fixed direction. In the case of the stars of high galactic concentration the rôle of such azimuth is played by the galactic longitude. Let $f(V, \alpha)dVd\alpha$ be the number of the observed stars with azimuths between α and $\alpha+d\alpha$ and with radial velocities between V and $V+dV$. The function $f(V, \alpha)$ is to be obtained from the lists of the radial velocity stars. If, further, $n(\alpha)d\alpha$ is the total number of the observed stars in the directions between α and $\alpha+d\alpha$, we have

$$n(\alpha)=\int_{-\infty}^{\infty} f(V, \alpha)dV.$$

Let $\phi(\xi, \eta)$ be the unknown frequency function of true velocities. According to the definition, $\phi(\xi, \eta)d\xi d\eta$ is the relative number of stars for which the velocity components fall within the limits ξ and $\xi+d\xi$, η and $\eta+d\eta$. We have

$$\int_{-\infty}^{\infty} d\eta\int_{-\infty}^{\infty}\phi(\xi, \eta)d\xi=1.$$

Among the observed stars with the azimuths confined within the interval $(\alpha, \alpha+d\alpha)$ we have $n(\alpha)d\alpha\phi(\xi, \eta)d\xi d\eta$ stars, which have the velocities confined within the element $d\xi d\eta$ of the "velocity plane" $\xi\eta$.

We may direct the ξ-axis to the azimuth $\alpha=0$. Then it is clear that all stars observed in the azimuth α for which the velocities lie within the strip S of $\xi\eta$ plane (see fig.) have the radial velocities lying between V and $V+dV$. Therefore from $n(\alpha)d\alpha$ stars within the interval $(\alpha, \alpha+d\alpha)$,

$$n(\alpha)d\alpha\int_{(S)}\phi(\xi, \eta)d\xi d\eta$$

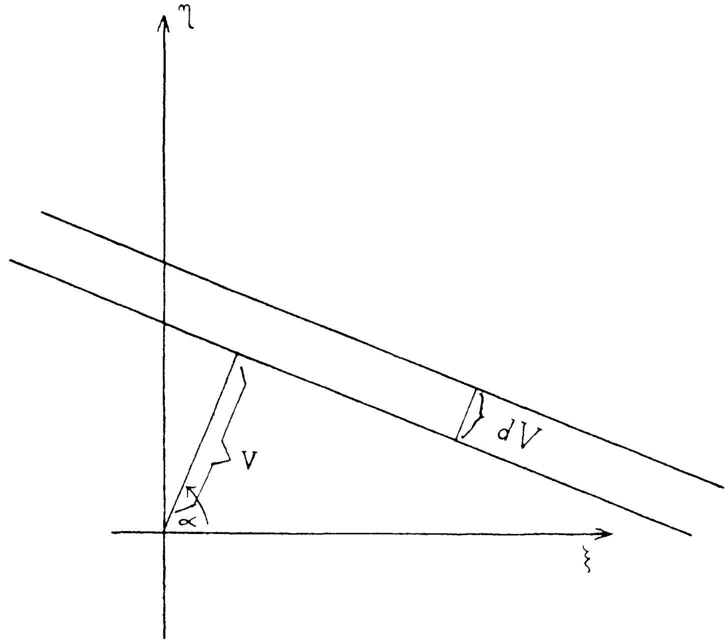
stars will have the radial velocities confined between V and $V+dV$. The integration is carried over the strip (S), perpendicular to the direction α and with the width dV.

On the other hand, we have denoted the number of such stars as

$$f(V, \alpha)dVd\alpha.$$

Therefore we have the equation

$$f(V, \alpha)dV = n(\alpha)\int_{(S)} \phi(\xi, \eta)d\xi d\eta. \qquad (1)$$



Let us introduce under the sign of the integral instead of ξ and η the new co-ordinates:

$$\xi' = \xi \cos \alpha + \eta \sin \alpha,$$
$$\eta' = -\xi \sin \alpha + \eta \cos \alpha.$$

It is clear that ξ' within the strip (S) varies between V and $V+dV$, and η' varies between $-\infty$ and $+\infty$. Therefore

$$\int_{(S)} \phi(\xi, \eta)d\xi d\eta = \int_V^{V+dV} d\xi' \int_{-\infty}^{\infty} \phi(\xi' \cos \alpha - \eta' \sin \alpha, \xi' \sin \alpha + \eta' \cos \alpha)d\eta'.$$

If we divide the equation (1) by $n(\alpha)$ and set

$$F(V, \alpha) = \frac{f(V, \alpha)}{n(\alpha)} = \frac{f(V, \alpha)}{\int f(V, \alpha)dV},$$

we obtain the equation

$$F(V, \alpha) = \int_{-\infty}^{\infty} \phi(V \cos \alpha - \eta' \sin \alpha, V \sin \alpha + \eta' \cos \alpha)d\eta'. \qquad (2)$$

The left-hand side of this equation may be obtained from the counts of stars in the radial-velocity lists.

Returning to the old co-ordinates ξ and η, we may write the equation (2) in the form

$$F(V, \alpha) = \int_{(L)} \phi(\xi, \eta) ds, \tag{3}$$

where the integration is carried over the straight line (L) defined by mean of equation

$$\xi \cos \alpha + \eta \sin \alpha = V, \tag{4}$$

and ds is the element of this line.

The equation (3) expresses the following problem :—

The value of the integral (3) *for every straight line of the* $\xi\eta$ *plane is given as the function of the parameters* V *and* α, *defining the straight line. The integrand* $\phi(\xi, \eta)$ *is to be found.*

The solution of this problem is comparatively simple. Let us introduce in both parts of (2) instead of V the expression

$$V = x \cos \alpha + y \sin \alpha + W, \tag{5}$$

where x, y and W are some arbitrary parameters. Then we can rewrite (2) in the form

$$F(x \cos \alpha + y \sin \alpha + W, \alpha) = \int_{-\infty}^{\infty} \phi(x \cos^2 \alpha + y \sin \alpha \cos \alpha + W \cos \alpha - \eta' \sin \alpha,$$
$$x \cos \alpha \sin \alpha + y^2 \sin^2 \alpha + W \sin \alpha + \eta' \cos \alpha) d\eta'.$$

If we introduce the new variable of integration

$$\eta' = U - x \sin \alpha + y \cos \alpha, \tag{6}$$

our equation takes the simple form

$$F(x \cos \alpha + y \sin \alpha + W, \alpha) = \int_{-\infty}^{\infty} \phi(x + W \cos \alpha - U \sin \alpha,$$
$$y + W \sin \alpha + U \cos \alpha) dU. \tag{7}$$

Multiplying both parts of this equation by $d\alpha$, integrating between 0 and 2π and changing the order of the integration on the right-hand side, we find

$$\int_0^{2\pi} F(x \cos \alpha + y \sin \alpha + W, \alpha) = \int_{-\infty}^{\infty} dU \int_0^{2\pi} \phi(x + W \cos \alpha - U \sin \alpha,$$
$$y + W \sin \alpha + U \cos \alpha) d\alpha. \tag{8}$$

Now it is easy to see that the integral

$$\Phi = \int_0^{2\pi} \phi(x + W \cos \alpha - U \sin \alpha, y + W \sin \alpha + U \cos \alpha) d\alpha, \tag{9}$$

depends only on x, y and $\sqrt{W^2 + U^2}$. In fact, if we introduce in (9)

$$\left. \begin{aligned} W &= G \cos \beta \\ U &= G \sin \beta \end{aligned} \right\}, \tag{10}$$

we obtain

$$\Phi = \int_0^{2\pi} \phi[x + G \cos (\alpha + \beta), y + G \sin (\alpha + \beta)] d\alpha,$$

and it is obvious that this integral depends only on x, y and $G=\sqrt{W^2+U^2}$ and is independent of $\beta\equiv\text{arctg}\frac{U}{W}$.

Therefore we may write simply:

$$\Phi(x, y, G)=\int_0^{2\pi}\phi(x+G\cos\alpha, y+G\sin\alpha)d\alpha \tag{11}$$

and

$$\int_0^{2\pi}F(x\cos\alpha+y\sin\alpha+W, \alpha)d\alpha=\int_{-\infty}^{\infty}\Phi(x, y, G)dU. \tag{12}$$

However,

$$dU=\frac{GdG}{\sqrt{G^2-W^2}},$$

and we may rewrite (12) in the form

$$\int_0^{2\pi}F(x\cos\alpha+y\sin\alpha+W, \alpha)d\alpha=2\int_W^{\infty}\Phi(x, y, G)\frac{GdG}{\sqrt{G^2-W^2}}. \tag{13}$$

This equation is an integral equation of Abel's type for the function $\Phi(x, y, G)$, and its solution is given by

$$\Phi(x, y, G)=-\frac{1}{\pi}\frac{1}{G}\frac{d}{dG}\int_G^{\infty}\frac{WdW}{\sqrt{W^2-G^2}}\int_0^{2\pi}F(x\cos\alpha+y\sin\alpha+W, \alpha)d\alpha. \tag{14}$$

We have, according to (11),

$$\Phi(x, y, 0)=2\pi\phi(x, y), \tag{15}$$

and we may rewrite (14) in the form

$$\phi(x, y)=-\frac{1}{2\pi^2}\lim_{G\to 0}\frac{1}{G}\frac{d}{dG}\int_G^{\infty}\frac{WdW}{\sqrt{W^2-G^2}}\bar{F}(x, y, W), \tag{16}$$

where the function

$$\bar{F}(x, y, W)=\int_0^{2\pi}F(x\cos\alpha+y\sin\alpha+W, \alpha)d\alpha \tag{17}$$

may be obtained from the observations. After some algebra we can reduce (16) to the form

$$\phi(x, y)=-\frac{1}{2\pi^2}\int_0^{\infty}\frac{1}{W}\frac{d\bar{F}(x, y, W)}{dW}dW. \tag{18}$$

This equation gives us the solution of our problem. The numerical calculation of $\bar{F}(x, y, W)$, when $F(V, \alpha)$ is given, may be carried out without difficulty.

We have actually applied our formulæ to the radial velocities of B-type stars observed in the galactic belt $\xi \mid b \mid < 20°$, and the results of this application are in satisfactory agreement with the velocity distribution derived from the direct counts of the known space velocities.

The details of this application will be given elsewhere.

The Three-dimensional Problem.—In the case of the three-dimensional problem we may derive from the catalogues the number of stars observed within the given solid angle $d\omega$ in the given direction, having the radial velocity confined within the limits V and $V+dV$. Let us denote this number by $f(V, l, b)d\omega$, where l and b are galactic longitude and latitude. If, further, $n(l, b)d\omega$ is the total number of observed stars in the same solid angle, we shall have

$$n(l, b) = \int f(V, l, b)dV. \tag{19}$$

As in the case of the two-dimensional problem we have the following relation between the frequency function of the space velocities $\phi(\xi, \eta, \zeta)$ and the observed function $f(V, l, b)$:—

$$f(V, l, b)dV = n(l, b)\iiint\limits_{(\Omega)}\phi(\xi, \eta, \zeta)d\xi d\eta d\zeta, \tag{20}$$

where the integration is extended over the volume (Ω) of the $\xi\eta\zeta$ space confined between two parallel planes, which are perpendicular to the direction (l, b) and have the distances V and $V+dV$ from the origin.

Dividing (20) by $n(l, b)$, we can bring this equation after some transformations to the form

$$F(V, l, b) = \iint\limits_{(\Sigma)}\phi(\xi, \eta, \zeta)d\sigma, \tag{21}$$

where

$$F(V, l, b) = \frac{f(V, l, b)}{n(l, b)}, \tag{22}$$

and the integration is extended on the plane (Σ), perpendicular to the direction (l, b) and distant V from the origin.

The equation of this plane is :

$$\xi \cos l \cos b + \eta \sin l \cos b + \zeta \sin b = V. \tag{Σ}$$

If we introduce the polar co-ordinates ρ and θ in the plane (Σ) with the origin at the point,

$$\xi = V \cos l \cos b\,; \qquad \eta = V \sin l \cos b\,; \qquad \zeta = V \sin b,$$

we shall have for the points of this plane

$$\begin{aligned}\xi &= V \cos l \cos b + \rho(\alpha_1 \cos \theta + \beta_1 \sin \theta),\\ \eta &= V \sin l \cos b + \rho(\alpha_2 \cos \theta + \beta_2 \sin \theta),\\ \zeta &= V \sin b + \rho(\alpha_3 \cos \theta + \beta_3 \sin \theta),\end{aligned}$$

where the coefficients α_1, α_2, α_3 and β_1, β_2, β_3 satisfy to the conditions

$$\begin{gathered}\alpha_1 \cos l \cos b + \alpha_2 \sin l \cos b + \alpha_3 \sin b = 0,\\ \beta_1 \cos l \cos b + \beta_2 \sin l \cos b + \beta_3 \sin b = 0.\\ \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0.\\ \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1\,; \qquad \beta_1^2 + \beta_2^2 + \beta_3^2 = 1.\end{gathered}$$

Now we can rewrite the equation (21) in these co-ordinates:

$$F(V, l, b) = \int_0^\infty \rho d\rho \int_0^{2\pi} \phi(V \cos l \cos b + \rho(\alpha_1 \cos \theta + \beta_1 \sin \theta), \\ V \sin l \cos b + \rho(\alpha_2 \cos \theta + \beta_2 \sin \theta), \\ V \sin b + \rho(\alpha_3 \cos \theta + \beta_3 \sin \theta))d\theta.$$

Integrating over all directions and changing the order of the integration on the right-hand side, we obtain

$$\int F(V, l, b)d\omega = \int \Phi \rho d\rho, \tag{23}$$

where

$$\Phi = \int d\omega \int_0^{2\pi} \phi(V \cos l \cos b + \rho(\alpha_1 \cos \theta + \beta_1 \sin \theta), \\ V \sin l \cos b + \rho(\alpha_2 \cos \theta + \beta_2 \sin \theta), \\ V \sin b + \rho(\alpha_3 \cos \theta + \beta_3 \sin \theta))d\theta.$$

If we introduce the new parameters

$$V = G\gamma_1 ; \qquad \rho \cos \theta = G\gamma_2 ; \qquad \rho \sin \theta = G\gamma_3 ; \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,$$

the integral Φ takes the form

$$\Phi = \int d\theta \int \phi(G(\gamma_1 \cos l \cos b + \gamma_2 \alpha_1 + \gamma_3 \beta_1), \\ G(\gamma_1 \sin l \cos b + \gamma_2 \alpha_2 + \gamma_3 \beta_2), \\ G(\gamma_1 \sin b + \gamma_2 \alpha_3 + \gamma_3 \beta_3))d\omega,$$

and it may be shown that it depends only on G. We may write

$$\Phi(G) = 2\pi \int \phi(G \cos l \cos b, G \sin l \cos b, G \sin b)d\omega ; \Phi(0) = 8\pi^2 \phi \mid 0, 0, 0) \tag{24}$$

We have now

$$G^2 = V^2 + \rho^2 ; \qquad \rho d\rho = GdG.$$

Therefore

$$\int F(V, l, b)d\omega = \int_V^\infty \Phi G dG \tag{25}$$

and

$$\Phi = -\frac{1}{V} \frac{d}{dV} \int F(V, l, b)d\omega. \tag{26}$$

Comparing (26) with (27) we find

$$\phi(0, 0, 0) = \frac{1}{8\pi^2} \Phi(0) = -\lim_{V \to 0} \frac{1}{V} \frac{d}{dV} \int F(V, l, b)d\omega.$$

Thus we may find $\phi(0, 0, 0)$. In the same way after some lengthy algebra we obtain

$$\phi(\xi, \eta, \zeta) = -\frac{1}{8\pi^2} \frac{1}{W} \frac{d}{dW} \int F(\xi \cos l \cos b + \eta \sin l \cos b + \zeta \sin b + W, l, b)d\omega.$$