POULKOVO OBSERVATORY
CIRCULAR

№ 4

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On the Temperatures of the Nuclei of planetary Nebulae

In the present paper a new method for the determination of the temperatures of the nuclei of planetary nebulae based on relative intensities of \( H \) and \( He^+ \) lines is proposed.

Our method makes use of the fact that for each pair of definite quantum states \((n, m)\) the ratio of Einstein's probability coefficients of spontaneous transition \( A_m^n \) for \( He^+ \), and \( A_m^n \) for \( H \) is constant and equal to 16 (See Appendix 1)

\[
\frac{A_m^n}{A_m^n} = 16 \quad \frac{}{}
\]

When estimating the temperatures of the nuclei of planetary nebulae Zanstra had supposed, that all quanta emitted from the star having frequencies greater than \( \nu_0 \) (the frequency of the limit of Lyman series) are absorbed by hydrogen atoms in nebulae. However, the nebulae contain so many atoms of \( He^+ \) that we can assume that the radiation beyond the series limit of \( He^+ \) \((4\nu_0)\) is absorbed by \( He^+ \) atoms (we neglect the line absorption). Therefore, only the radiation with frequencies between \( \nu_0 \) and \( 4\nu_0 \) is absorbed by hydrogen. Let the number of quanta with \( \nu > 4\nu_0 \) be \( N'_{ul} \), and with \( \nu_0 < \nu < 4\nu_0 \) be \( N_{ul} \). According to Planck's law

\[
\frac{N'_{ul}}{N_{ul}} = \frac{\int_{4\nu_0}^{\infty} \frac{x^2}{e^{\frac{x^2}{kT}} - 1} dx}{\int_{\nu_0}^{4\nu_0} \frac{x^2}{e^{\frac{x^2}{kT}} - 1} dx} \quad x_0 = \frac{\hbar \nu_0}{kT} \quad \frac{}{}
\]

where \( T \) is the surface temperature of the star, \( \hbar \) and \( k \) have their usual meaning.

At the same time \( N'_{ul} \) and \( N_{ul} \) are the numbers of ionizations and recombinations of \( He^+ \) and \( H \) atoms per second. Let further

\[
p_1 N_{ul}, p_2 N_{ul}, p_3 N_{ul}, \quad \frac{}{}
\]

be the numbers of recombinations of \( H \) per second, by which the free electron jumps immediately into 1st, 2nd, 3rd quantum states.
Let
\[ p'_1 N'_w, p'_2 N'_w, p'_3 N'_w, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3') \]
be the corresponding numbers for He\(^+\).

We have
\[ \sum_i p_i = 1; \quad \sum_i p'_i = 1. \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4), \]
and neglecting the variation of \( p_i \) corresponding to the variation of temperature (See Appendix II).
\[ p_i = p'_i; \quad (i = 1, 2, 3, \ldots) \ldots \ldots \ldots \ldots \ldots \ldots \ldots (5). \]

The number \( \mathfrak{R}_{nm} \) of each type of atomic transitions \( n \to m \) per second is a linear homogeneous function of \( p_i N'_{wi} \) (J. A. Carroll, M. N., 90, 588, 1930, our \( \mathfrak{R}_{nm} \) have the same meaning as Carroll's \( A_{nm} N_n \)).
\[ \mathfrak{R}_{nm} = \sum_i P'_{nm} p_i N'_{wi}; \quad \mathfrak{R}'_{nm} = \sum_i P'_{nm} p'_i N'_w. \ldots \ldots \ldots \ldots \ldots \ldots \ldots (6). \]

The coefficients \( P'_{nm} \) are homogeneous functions of degree 0 of Einstein's coefficient \( A_{nm} \) (Carroll's \( N_n \) are homogeneous functions of degree \(-1\)). This function is the same for \( H, \text{He}^+ \) and for any atom with a homologous spectrum. From Euler's theorem and (1) we conclude
\[ P'_{nm} = P_{nm} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (7). \]

From the equations (6) and (7) we have
\[ \frac{\mathfrak{R}'_{nm}}{\mathfrak{R}_{nm}} = \frac{N'_w}{N_w} \ldots \ldots \ldots \ldots \ldots \ldots \ldots (8). \]

Applying this equation to the transition \( 4 \to 2 \), we have
\[ \frac{\mathfrak{R}'_{42}}{\mathfrak{R}_{42}} = \frac{N'_w}{N_w} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (8'). \]

According to the definition of Einstein's probability coefficient we have
\[ \mathfrak{R}'_{42} = N'_4 A'_{42}; \quad \mathfrak{R}'_{42} = N'_4 A'_{42} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (9), \]
where \( N'_4 \) is the number of He\(^+\) atoms in 4th quantum state. Hence
\[ \mathfrak{R}'_{42} = \frac{A_{48}}{A_{48}} \mathfrak{R}'_{42} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (10). \]

Introducing (10) into (8') we obtain
\[ \frac{\mathfrak{R}_{42}}{\mathfrak{R}_{42}} = \frac{A_{48}}{A_{48}} \frac{\mathfrak{R}'_{42}}{\mathfrak{R}_{42}} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (11). \]
In the absence of absorption the ratio of intensities of 4686 and $H_{\beta}$ is

$$\frac{J'_{48}}{J_{48}} = \frac{N'_{48}}{N_{48}} \frac{\lambda'_{48}}{\lambda_{48}} = \frac{N'_{48}}{N_{48}} \frac{\lambda_{48}}{\lambda'_{48}}$$

(12),

where $\lambda$'s are the wavelengths of the corresponding emission.

From (11) and (12) we find

$$\frac{N'_{ul}}{N_{ul}} = \frac{A_{48}}{A'_{48}} \frac{\lambda'_{48}}{\lambda_{48}} \frac{J'_{48}}{J_{48}}.$$  

(13).

The ratio $\frac{J'_{48}}{J_{48}}$ may be obtained from measurements of intensities of 4686 and $H_{\beta}$. Therefore we also have the value $\frac{N'_{ul}}{N_{ul}}$ from observation. The following Table 1 contains the values of $\frac{N'_{ul}}{N_{ul}}$ for different values of $T$. The table is calculated according to (2). Taking $\frac{N'_{ul}}{N_{ul}}$ from observations we find $T$ from this table.

### Table 1

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>$T$</th>
<th>$\frac{N'<em>{ul}}{N</em>{ul}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>0.2</td>
<td>786.000°</td>
<td>9</td>
</tr>
<tr>
<td>0.3</td>
<td>623.000°</td>
<td>4.5</td>
</tr>
<tr>
<td>0.6</td>
<td>262.000°</td>
<td>1</td>
</tr>
<tr>
<td>1.0</td>
<td>187.000°</td>
<td>0.3</td>
</tr>
<tr>
<td>1.5</td>
<td>105.000°</td>
<td>0.075</td>
</tr>
<tr>
<td>2.0</td>
<td>78.500°</td>
<td>0.02</td>
</tr>
<tr>
<td>2.5</td>
<td>69.800°</td>
<td>0.005</td>
</tr>
</tbody>
</table>

Applications to the planetary nebulae NGC 7009 and 7027.

In the second column of table 2 the ratio $\frac{4686}{H_{\beta}}$ is given for two nebulae according to Berman's observations (L. O. B. 15, 86).

### Table 2

<table>
<thead>
<tr>
<th>Nebulae</th>
<th>$\frac{4686}{H_{\beta}}$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7009</td>
<td>2°18</td>
<td>115.000°</td>
</tr>
<tr>
<td>7027</td>
<td>1°03</td>
<td>165.000°</td>
</tr>
</tbody>
</table>
The test of the method. The general form of the equation (13) is

\[ \frac{N''_{n\ell}}{N_{n\ell}} = \frac{A_{nm}}{A_{nk}} \frac{\lambda'_{nk}}{\lambda_{nm}} J'_{nk} \]

(14).

Applying this equation to the lines: 4686 and \( H_{\alpha} \), 4542 and \( H_{\eta} \), 4200 and \( H_{\epsilon} \), we obtain

\[ \frac{A_{42}}{A'_{42}} \frac{\lambda'_{42}}{\lambda_{42}} J'_{42} = \frac{A_{94}}{A'_{94}} \frac{\lambda'_{94}}{\lambda_{94}} J'_{94} = \frac{A_{11.4}}{A'_{11.4}} \frac{\lambda'_{11.4}}{\lambda_{11.4}} J'_{11.4} \]

For the pairs under consideration the factors \( \frac{A_{nm}}{A'_{nk}} \frac{J'_{nk}}{J_{nm}} \) are nearly the same. Therefore:

\[ \frac{J'_{42}}{J_{42}} = \frac{J'_{94}}{J_{94}} = \frac{J'_{11.4}}{J_{11.4}} \]

According to Berman, we have correspondingly the following differences of brightness for the nebula NGC 7027:

\[ 1^\circ 03, \quad 0^\circ 86, \quad 0^\circ 87 \]

The deviations from mean value 0^\circ 91 are comparatively small, and thus the agreement of the theory with observations seems to be very satisfactory.

Appendix I. From the Schrödinger's equation for \( H \) and \( He^+ \) atoms it follows, that the normalized eigenfunctions \( \phi'_n \) of \( He^+ \) are connected with the corresponding eigenfunctions \( \phi_n \) of \( H \) according to the relation

\[ \phi'_n (x, y, z) = \sqrt{8} \phi_n (2x, 2y, 2z) \]

The matrix elements \( q'_{nm} \) of \( He^+ \) atom are

\[ q'_{nm} = \int \int \int \phi'_n \bar{\phi}_m \rho(x, y) dx dy dz = \frac{1}{\sqrt{2}} \int \int \int \phi_n (\xi, \eta, \zeta) \bar{\phi}_m (\xi, \eta, \zeta) \xi d\xi d\eta d\zeta = \frac{1}{2} q_{nm}. \]

where \( q_{nm} \) are the corresponding matrix elements of \( H \) atom.

The Einstein probability coefficient for \( H \) atom has the form:

\[ A_{nm} = C_{\nu} |q_{nm}|^2 \]

and for \( He^+ \) atom

\[ A'_{nm} = C'_{\nu'} |q'_{nm}|^2 \]

Since \( \nu' = 4\nu \) and \( q'_{nm} = \frac{1}{2} q_{nm} \), we may write

\[ A'_{nm} = 16 C'_{\nu} |q_{nm}|^2 = 16 A_{nm}. \]
Appendix II. Applying (1) to the probabilities of the binding of a free electron with energy $\varepsilon$ in a particular quantum state $m$, we have

$$A_m' (\varepsilon) = 16 A_m \left( \frac{\varepsilon}{4} \right).$$

Therefore the ratio of the number of transitions of free electrons into the state $m$ to the number of all recombinations is given by

$$p_m' (T) = \frac{p_m h^2}{N_u} = \frac{\int A_m' (\varepsilon) e^{-\frac{\varepsilon}{kT}} d\varepsilon}{\sum m \int A_m' (\varepsilon) e^{-\frac{\varepsilon}{kT}} d\varepsilon} = \frac{\int A_m \left( \frac{\varepsilon}{4} \right) e^{-\frac{\varepsilon}{kT}} d\varepsilon}{\sum m \int A_m \left( \frac{\varepsilon}{4} \right) e^{-\frac{\varepsilon}{kT}} d\varepsilon}$$

if the velocity distribution of electrons is in accordance with Maxwell's law.

Introducing $\frac{\varepsilon}{4} = \varepsilon'$ we find

$$p_m' (T) = \frac{\int A_m (\varepsilon') e^{-\frac{4\varepsilon'}{kT}} d\varepsilon'}{\sum m \int A_m (\varepsilon') e^{-\frac{4\varepsilon'}{kT}} d\varepsilon'} = p_m \left( \frac{T}{4} \right).$$

Neglecting the variation of $p_m$ with temperature, we may put

$$p_m' = p_m$$

V. Ambarzumian.