THE RADIATIVE EQUILIBRIUM OF A PLANETARY NEBULA

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The purpose of the present paper is the consideration of the radiative equilibrium of planetary nebulæ. Recent investigations on the excitation of nebular radiation have shown that the nebular material cannot be treated simply as a "grey body," and that therefore the interaction of radiation of each wave-length with nebular material should be the subject of detailed investigation. It might seem that the introduction of radiation of many wave-lengths into the problem would so enormously increase the difficulties that it would be hopeless to try to solve it. But we shall find that the problem allows many simplifications, which are connected with the physical properties of the model and cannot disturb the adequacy of it.

The nebula is supposed to consist of hydrogen, for which all atomic constants, including the transition probabilities, are known. But, with slight modifications, the method developed here is applicable to nebulæ composed of other substances. It is certain that hydrogen is one of the most important constituents of gaseous nebulæ, and our results, corrected for the existence of other substances, will be applicable also to the real planetary nebulæ.

A method of the reduction of the spherical problem to a plane problem, proposed by Professor Milne,* is adopted in the present paper.

According to the theory of nebular luminosity developed by Zanstra, all or almost all quanta emitted from the central star having frequencies greater than ν_0 (the frequency of the limit of the Lyman series) are absorbed by the hydrogen atoms in the nebula. As was shown by Zanstra, the nebula, in the place of each absorbed quantum having a frequency greater than ν_0, creates an L_α-quantum, where L_α is the first line of the Lyman series of hydrogen. For the sake of completeness we will repeat here Zanstra's line of argument. For simplicity we will call the radiation beyond the head of the Lyman series briefly "ultra-violet radiation," and the corresponding quanta "ultra-violet quanta."

We shall consider the various transformations of an ultra-violet quantum, which is emitted from the surface of the central star. It will be absorbed by the nebular envelope, and this absorption will be accompanied by the ionisation of a hydrogen atom. After an interval of time the freed electron is captured again by a proton. There are two possibilities by this capture: I, the electron jumps immediately into the deepest level $1S$ (first level); and II, the electron jumps into one of the excited levels. In case I a

* Milne, *Zs. für Astrophysik*, **1**, 98, 1930.

new ultra-violet quantum is emitted and the initial state is restored. In case II the electron makes a chain of transitions, the last link of which will be a transition into the first level. The dilution of radiation is so great and the density of matter is so low, that the interruption of these transitions is very improbable. The last transition into the normal state is accompanied by emission of a quantum of the Lyman series. There are again two possibilities: (*a*) An L_α-quantum is emitted. Zanstra's theory requires that the optical thickness of the nebula in ultra-violet light is at least of the order of unity. But the coefficient of the line-absorption in the Lyman series is some thousand times larger than the absorption coefficient beyond the head of the series. The emitted L_α-quantum, therefore, will be absorbed by a hydrogen atom in the normal state. This atom passes into the second level, and then, owing to the absence of external perturbations during its short lifetime, turns back to the first level, emitting again an L_α-quantum. Thus the L_α-quantum remains unchanged, and we may say that it suffers only the scattering processes. These processes may be repeated many times until the quantum reaches the outer boundary of nebula and flies away. (*b*) A quantum of some other line of the Lyman series is emitted. For simplicity we assume that it is an L_β-quantum. In this line the optical thickness of the nebula is also very large and the emitted quanta will be absorbed. This absorption is accompanied by the transition of an atom from the first to the third level. The atom in the third level has two possibilities: either it makes the transition of the type $3 \rightarrow 2 \rightarrow 1$, emitting the quanta H_α and L_α successively, or it passes immediately into the first level, emitting again L_β. In the first case the final product is an L_α-quantum. Its further fate is described above. In the second case the quantum L_β will be absorbed, and there thus exists a finite probability of creation of the quantum L_α. After many absorptions and re-emissions the probability of survival of an L_β-quantum will be very small, and the probability of creation of an L_α-quantum will be practically equal to unity.

In this manner in both cases (*a*) and (*b*) the final product is an L_α-quantum. It is easily seen that our consideration may be generalised to the cases when the transitions are accompanied, instead of by the emission of an L_β-quantum, by the emission of a quantum L_γ, L_δ, etc. Let p_1 be the probability of the case I and $1-p$ the probability of the case II. Our considerations may be summed up in the following manner.

After the absorption of an ultra-violet quantum there is a finite probability p of the re-emission of it with the same wave-length and a finite probability $1-p$ of the re-emission of the quantum L_α. We will not take into account the intermediate stages in which the absorbed quantum may appear as L_β-, L_γ-, quanta, etc. These will have little influence on our results. The quanta L_α cannot suffer new transformations and can only be scattered.

The problem in this way is reduced to the study of two superposed fields of radiation: the field of ultra-violet quanta and the L_α-radiation field in the planetary nebulæ. We shall consider first the field of ultra-violet quanta.

The Field of Ultra-violet Quanta.—As mentioned above, we shall use the method of the reduction of the spherical problem to a plane problem, developed by Professor Milne. Let k be the absorption coefficient of the ultra-violet radiation per atom. This coefficient depends upon the wave-length. We shall take its mean value. Let, further, n be the number of H atoms in the first level in 1 cm.^{3}, r_1 and r_2 the distances of the inner and outer boundary of the nebular ring from the central star. Then the optical depth at the distance r from the central star is

$$\tau = \int_r^{r_2} nkdr. \tag{1}$$

The equation of transfer of energy of ultra-violet quanta may be written, according to Milne, in the form

$$\frac{1}{2}\frac{dI(\tau)}{d\tau} = I(\tau) - B(\tau), \tag{2}$$

$$\frac{1}{2}\frac{dI'(\tau)}{d\tau} = B(\tau) - I'(\tau), \tag{3}$$

if we use an approximation of the Schwarzschild-Schuster type.

Here $I(\tau)$ is the average intensity of the diffuse ultra-violet radiation of the nebula in the outward direction at the point τ, and $I'(\tau)$ is the average intensity of the same radiation at the same point in the inward direction. The quantity $4\pi B(\tau)d\tau$ is the amount of energy of ultra-violet quanta emitted in the layer $d\tau$ per second. The same layer absorbs the diffuse ultra-violet radiation from various parts of the nebular ring. The absorbed energy is equal to $2\pi[I(\tau)+I'(\tau)]d\tau$. Besides this, the layer absorbs the radiation of the central star. Let πS be the amount of ultra-violet energy falling on each square centimetre of the inner surface of the nebula. At the point τ this amount is reduced to $\pi S e^{-(\tau_1-\tau)}$, where

$$\tau_1 \equiv \int_{r_1}^{r_2} nkdr \tag{4}$$

is the optical thickness of the nebula. From this amount our layer absorbs $\pi S e^{-(\tau_1-\tau)}d\tau$.

Since from the quanta absorbed only the fraction p is re-emitted again as ultra-violet quanta, the equation of radiative equilibrium may be written in the form:

$$p[2\pi(I+I') + \pi S e^{-(\tau_1-\ )}] = 4\pi B(\tau),$$

or

$$p[I(\tau)+I'(\tau)+\tfrac{1}{2}S e^{-(\tau_1-\ )}] = 2B(\tau). \tag{5}$$

Introducing the boundary conditions *

$$I'(0) = 0, \qquad I(\tau_1) = I'(\tau_1), \tag{6}$$

we take account of the diffuse radiation incident on any portion of the inner

* Milne, *Zs. für Astrophysik*, 1, 105, 1930.

face of the nebular shell and arriving from other portions of the inner face.

From the equations (2) and (3) we have

$$\tfrac{1}{2}\frac{d(I+I')}{d\tau}=I-I' \tag{7}$$

$$\tfrac{1}{2}\frac{d(I-I')}{d\tau}=I+I'-2B. \tag{8}$$

Differentiating (7) and comparing with (8) we obtain

$$\tfrac{1}{4}\frac{d^2(I+I')}{d\tau^2}=I+I'-2B. \tag{9}$$

Introducing (5) in (9) we find the following differential equation for $I+I'$:

$$\tfrac{1}{4}\frac{d^2(I+I')}{d\tau^2}=(1-p)(I+I')-\frac{p}{2}Se^{-(\tau_1-\tau)}. \tag{10}$$

The general solution of this equation is

$$I+I'=Ae^{-\lambda\tau}+Be^{\lambda\tau}+\frac{2p}{3-4p}Se^{-(\tau_1-\tau)}, \tag{11}$$

where A and B are constants of integration and $\lambda=2\sqrt{1-p}$. Introducing (11) in (5) we find :

$$B(\tau)=\frac{p}{2}\left(Ae^{-\lambda\tau}+Be^{\lambda\tau}+\frac{3}{2(3-4p)}Se^{-(\tau_1-\tau)}\right). \tag{12}$$

Introducing (11) in (7) we obtain

$$I(\tau)-I'(\tau)=-\frac{\lambda}{2}Ae^{-\lambda\tau}+\frac{\lambda}{2}Be^{\lambda\tau}+\frac{p}{3-4p}Se^{-(\tau_1-\tau)}. \tag{13}$$

Adding and subtracting (11) and (13) we find $I(\tau)$ and $I'(\tau)$:

$$I(\tau)=\tfrac{1}{2}\left(1-\frac{\lambda}{2}\right)Ae^{-\lambda\tau}+\tfrac{1}{2}\left(1+\frac{\lambda}{2}\right)Be^{\lambda\tau}+\frac{3p}{2(3-4p)}Se^{-(\tau_1-\tau)}, \tag{14}$$

$$I'(\tau)=\tfrac{1}{2}\left(1+\frac{\lambda}{2}\right)Ae^{-\lambda\tau}+\tfrac{1}{2}\left(1-\frac{\lambda}{2}\right)Be^{\lambda\tau}+\frac{p}{2(3-4p)}Se^{-(\tau_1-\tau)}. \tag{15}$$

The first of the conditions (6) we may write according to (15) in the form :

$$A\left(1+\frac{\lambda}{2}\right)+B\left(1-\frac{\lambda}{2}\right)+\frac{p}{3-4p}S=0. \tag{16}$$

The second of the conditions (6) gives us :

$$\lambda Be^{\lambda\tau_1}+\frac{2p}{3-4p}S=\lambda Ae^{-\lambda\tau_1}. \tag{17}$$

From the equations (16) and (17) we find the following values of coefficients A and B :

$$A=\frac{\left(1-\frac{\lambda}{2}\right)e^{\tau_1}-\frac{\lambda}{2}e^{\lambda\tau_1}}{\frac{\lambda}{2}\left[\left(1-\frac{\lambda}{2}\right)e^{-\lambda\tau_1}+\left(1+\frac{\lambda}{2}\right)e^{\lambda\tau_1}\right]}\frac{p}{3-4p}Se^{-\tau_1}, \tag{18}$$

$$B=-\frac{\left(1+\frac{\lambda}{2}\right)e^{\tau_1}+\frac{\lambda}{2}e^{-\lambda\tau_1}}{\frac{\lambda}{2}\left[\left(1-\frac{\lambda}{2}\right)e^{-\lambda\tau_1}+\left(1+\frac{\lambda}{2}\right)e^{\lambda\tau_1}\right]}\frac{p}{3-4p}Se^{-\tau_1}. \tag{19}$$

These values of A and B introduced in (12), (14) and (15) give us the solution for the field of the ultra-violet quanta.

To compare this solution with the case when the diffuse radiation in (5) is neglected we shall consider a numerical example. For the representation of the solution in numerical form it is necessary to know τ_1 and p. Probably the values of τ_1 are different for different nebulæ. It seems that $\tau_1=2$ will be an appropriate value for some nebulæ. The value of p is to be calculated from pure physics. Cillié * has computed the relative probabilities of capture of electrons by protons on the different levels. From his results we have deduced the fraction of captured electrons which pass immediately from the free state to the first level, re-emitting the ultra-violet quanta. This fraction is our p. The value of p depends on the temperature of free electrons. For the different temperatures we have :

T	10,000°	20,000°	50,000°
p	0·46	0·49	0·57

Putting in (12) $\tau_1=2$ and $p=0\cdot5$, we have the following numerical solution for $B(\tau)$:

$$B(\tau)=\tfrac{1}{4}S[\tfrac{3}{2}e^{-(\tau_1-\tau)}-0\cdot032e^{-1\cdot4\tau}-0\cdot044e^{1\cdot4\tau}].$$

When the diffuse radiation in (5) is neglected we find :

$$B(\tau)=\tfrac{1}{8}Se^{-(\tau_1-\tau)}.$$

If $\tau=\tau_1$, the first of these formulæ gives : $B(\tau_1)=0\cdot194S$; the second, $B(\tau_1)=0\cdot125S$. Thus the difference is of the same order as $B(\tau_1)$.

The Field of L_a-radiation.—For the study of the field of L_a-radiation we shall introduce the absorption-coefficient κ within the line L_a per hydrogen atom in the normal state. The optical depth for this line is defined by

$$t=\int_r^{r_2}n\kappa dr. \tag{20}$$

The ratio $\frac{\kappa}{k}=\omega$ may be supposed constant when the temperature variations within the nebula are neglected. In fact, k is a function of atomic

* Cillié, *M.N.*, **92**, 820, 1932.

constants and of the Doppler breadth of the line only. This breadth depends upon the temperature. When $\kappa/k=\omega$ is constant, the ratio t/τ is also constant, and we have

$$\frac{t}{\tau}=\frac{\kappa}{k}=\omega. \tag{21}$$

If the temperature of the nebula is of the order 10^3-10^4 degrees, the quantity ω will be also of the order 10^3-10^4. Since we have supposed that the optical thickness τ_1 of the nebula in the ultra-violet region is of the order of unity and larger, the optical thickness in the line L_α,

$$t_1=\int_{r_1}^{r_2}\kappa n dr,$$

will be of the order 10^3-10^4 or larger.

The equations of transfer of the radiation in the line L_α have the same form as (2) and (3). Let $K(t)$ be the average intensity of the diffuse L_α-radiation of nebula in the outward direction at the point t, and $K'(t)$ the average intensity of the same radiation at the same point in the inward direction. The equations of transfer are :

$$\tfrac{1}{2}\frac{dK(t)}{dt}=K(t)-C(t), \tag{22}$$

$$\tfrac{1}{2}\frac{dK'(t)}{dt}=C(t)-K'(t), \tag{23}$$

where $4\pi C(t)dt$ is the amount of energy emitted by the layer dt in the line L_α per second. This layer absorbs the diffuse L_α-radiation from the other parts of nebula. The quantity of diffuse radiation absorbed is $2\pi[K(t)+K'(t)]dt$. The number of L_α-quanta emitted by the central star is negligible, since the number of ultra-violet quanta transformed into L_α-quanta is some thousand times larger. Moreover, the quanta L_α emitted by the central star are absorbed in the deepest layers of the nebula. The number of ultra-violet quanta which are absorbed in the layer dt transformed into L_α-quanta is

$$\frac{(1-p)[2\pi(I+I')+\pi S e^{-(\tau_1-\tau)}]d\tau}{h\nu_c},$$

where ν_c is the average frequency of ultra-violet radiation.

Thus the L_α-radiation created in dt according to (5) is

$$\frac{(1-p)}{p}\frac{\nu_\alpha}{\nu_c}4\pi B(\tau)d\tau=\frac{1-p}{p}\frac{\nu_\alpha}{\nu_c}4\pi B(\tau)\frac{dt}{\omega},$$

since the amount of energy of each L_α-quantum is $h\nu_\alpha$. Hence the equation of radiative equilibrium is

$$4\pi C(t)dt=2\pi[K(t)+K'(t)]dt+4\pi\frac{1-p}{p}\frac{\nu_\alpha}{\nu_c}B(\tau)\frac{dt}{\omega},$$

or

$$C(t)=\tfrac{1}{2}[K(t)+K'(t)]+\frac{\nu_a}{\nu_c}\frac{1-p}{p\omega}B(\tau). \tag{24}$$

The boundary conditions are :

$$K'(0)=0\ ; \qquad K'(t_1)=K(t_1). \tag{25}$$

From the equations (22) and (23) we have

$$\tfrac{1}{2}\frac{d(K+K')}{dt}=K-K', \tag{26}$$

$$\tfrac{1}{2}\frac{d(K-K')}{dt}=K+K'-2C(t). \tag{27}$$

Differentiating (26) and introducing (27) we find :

$$\tfrac{1}{4}\frac{d^2(K+K')}{dt^2}=K+K'-2C(t), \tag{28}$$

or according to (24),

$$\tfrac{1}{4}\frac{d^2(K+K')}{dt^2}=-\frac{2\nu_a}{\nu_c}\frac{1-p}{p\omega}B(\tau). \tag{29}$$

Writing $B(\tau)$ in the form

$$B(\tau)=\frac{p}{2}\left(Ae^{-\lambda\tau}+Be^{\lambda\tau}+De^{-(\tau_1-\tau)}\right)=\frac{p}{2}\left(Ae^{-\frac{\lambda}{\omega}t}+Be^{\frac{\lambda}{\omega}t}+De^{\frac{-(t_1-t)}{\omega}}\right), \tag{30}$$

where

$$D=\frac{3}{2(3-4p)}S, \tag{31}$$

we find the following solution of equation (29) :

$$K(t)+K'(t)=a+bt-\frac{8\nu_a}{\nu_c}\frac{1-p}{2\lambda^2}\omega\left(Ae^{-\frac{\lambda}{\omega}t}+Be^{\frac{\lambda}{\omega}t}+D\lambda^2e^{-\frac{t_1-t}{\omega}}\right),$$

where a and b are constants of integration.

Differentiating this expression we find, according to (26),

$$K(t)-K'(t)=\frac{b}{2}-\frac{4\nu_a}{\nu_c}\frac{1-p}{2\lambda}\left(-Ae^{-\frac{\lambda}{\omega}t}+Be^{\frac{\lambda}{\omega}t}+D\lambda e^{-\frac{t_1-t}{\omega}}\right).$$

According to the definition of λ,

$$\lambda=2\sqrt{1-p}.$$

Therefore

$$K(t)+K'(t)=a+bt-\frac{\nu_a}{\nu_c}\omega\left(Ae^{-\frac{\lambda}{\omega}t}+Be^{\frac{\lambda}{\omega}t}+D\lambda^2e^{-\frac{t_1-t}{\omega}}\right), \tag{32}$$

$$K(t)-K'(t)=\frac{b}{2}-\frac{\nu_a}{\nu_c}\frac{\lambda}{2}\left(-Ae^{-\frac{\lambda}{\omega}t}+Be^{\frac{\lambda}{\omega}t}+D\lambda e^{-\frac{t_1-t}{\omega}}\right). \tag{33}$$

From (32) and (33) we have

$$K(t)=\frac{a}{2}+\frac{b}{4}+\frac{b}{2}t-\frac{\nu_a}{2\nu_c}\omega\left[A\left(1-\frac{\lambda}{2\omega}\right)e^{-\frac{\lambda}{\omega}t}+B\left(1+\frac{\lambda}{2\omega}\right)e^{+\frac{\lambda}{\omega}t}+D\lambda^2\left(1+\frac{1}{2\omega}\right)e^{-\frac{t_1-t}{\omega}}\right] \quad (34)$$

$$K'(t)=\frac{a}{2}-\frac{b}{4}+\frac{b}{2}t-\frac{\nu_a}{2\nu_c}\omega\left[A\left(1+\frac{\lambda}{2\omega}\right)e^{-\frac{\lambda}{\omega}t}+B\left(1-\frac{\lambda}{2\omega}\right)e^{+\frac{\lambda}{\omega}t}+D\lambda^2\left(1-\frac{1}{2\omega}\right)e^{-\frac{t_1-t}{\omega}}\right] \quad (35)$$

The first of the conditions (25) reduces to

$$\frac{a}{2}-\frac{b}{4}-\frac{\nu_a}{2\nu_c}\omega\left[A\left(1+\frac{\lambda}{2\omega}\right)+B\left(1-\frac{\lambda}{2\omega}\right)+4D(1-p)\left(1-\frac{1}{2\omega}\right)e^{-\frac{t_1}{\omega}}\right]=0, \quad (36)$$

and the second,

$$\frac{b}{4}-\frac{\nu_a}{2\nu_c}\omega\left[A\left(1-\frac{\lambda}{2\omega}\right)e^{-\frac{\lambda}{\omega}t_1}+B\left(1+\frac{\lambda}{2\omega}\right)e^{\frac{\lambda t_1}{\omega}}+4D(1-p)\left(1+\frac{1}{2\omega}\right)\right]$$

$$=-\frac{b}{4}-\frac{\nu_a}{2\nu_c}\omega\left[A\left(1+\frac{\lambda}{2\omega}\right)e^{-\frac{\lambda}{\omega}t_1}+B\left(1+\frac{\lambda}{2\omega}\right)e^{\frac{\lambda}{\omega}t_1}+4D(1-p)\left(1-\frac{1}{2\omega}\right)\right],$$

or

$$b=\frac{\nu_a}{\nu_c}[-\lambda Ae^{-\lambda\tau_1}+\lambda Be^{\lambda\tau_1}+4D(1-p)]. \quad (37)$$

Introducing (37) in (36) we find:

$$a=\frac{\nu_a}{2\nu_c}[-\lambda Ae^{-\lambda\tau_1}+\lambda Be^{\lambda\tau_1}+4D(1-p)]$$
$$+\frac{\nu_a}{2\nu_c}\omega\left[A\left(1+\frac{\lambda}{2\omega}\right)+B\left(1-\frac{\lambda}{\omega}\right)+4D(1-p)\left(1-\frac{1}{2\omega}\right)e^{-\tau_1}\right]. \quad (38)$$

The equations (34) and (35) together with (37) and (38) give the solution for the L_a-field.

The Density of Radiation in the Inner Layers of the Nebula.—We have denoted above by πS the amount of energy of the ultra-violet radiation falling on each square centimetre of the inner surface of the nebula from the star. In the absence of re-emission the mean intensity of the ultra-violet radiation in this region will be equal to $\frac{\pi S}{4\pi}=0{\cdot}25S$. In the case when re-emission is taken into account, the average intensity of the radiation increases and is equal to $\frac{1}{4}S+\frac{1}{2}(I_1+I_2)$. In the example considered above, the mean intensity of ultra-violet radiation at $\tau=\tau_1$ reaches the value $0{\cdot}39S$. We may conclude that though the re-emission increases the density of the ultra-violet radiation (which is proportional to the average intensity), this density remains of the same order as in the absence of re-emission. The state of affairs entirely changes when we consider the L_a-field. Owing to the large optical thickness of the nebula in the L_a-line, and to the fact that all L_a-quanta absorbed are re-emitted in the same frequency, the density of L_a-radiation in the inner layers of the ring is very large. In fact, in the numerical example considered above, the average intensity of L_a-radiation, $\frac{1}{2}(K_1+K_2)$ at $t=t_1$ according to (37) is equal to $12,000S$, where πS is again the energy of the whole ultra-violet radiation falling on each square centimetre of the inner surface of the nebula from the central star.* Therefore the density of L_a-radiation in the example considered is 48,000 times larger than the density of the whole diluted ultra-violet radiation of the central star. The rough estimate shows that the ultra-violet radiation of the black body at the temperatures of the order 40,000°–50,000° is about $5 \cdot 10^4$ times stronger than the same radiation within the Doppler width of the L_a-line corresponding to the temperature of the nebular matter. Thus the density of L_a-radiation in the inner layers of the nebular ring will be $48,000 \times 5 \cdot 10^4 = 2 \cdot 7 \times 10^9$ times larger than the density of diluted radiation of the central star within the line-width. If the dilution factor is of the order 10^{-13}, the density of diluted radiation within the width of the line will be of the order $10^{-13}\rho_a$ where ρ_a is the density of the black-body radiation in the same frequency interval at the temperature of the central star. Then the density of diffuse L_a-radiation of the nebula in the inner regions of the ring will be of the order

$$27 \times 10^8 \times 10^{-13}\rho_a = 0 \cdot 27 \times 10^{-3}\rho_a.$$

Absorbing an L_a-quantum, the hydrogen atom from the state $1S$ passes into the state $2P$. With such a large density of radiation the excitation will be considerable. If n_2 is the number of H-atoms in the $2P$ state in one cm.^{3}, we shall have

$$n_2/n_1 = 5 \times 10^{-5} \tag{39}$$

when the temperature of the central star is 40,000°.

It is known that the absorption coefficient beyond the limit of the Lyman series is of the order $0 \cdot 5 \times 10^{-17}$. The nebular shell with optical thickness $\tau = 2$ will contain 4×10^{17} atoms of hydrogen per square centimetre. By application of (39) we find that the number of hydrogen atoms per square centimetre in the state $2P$ will be of the order 2×10^{13}. But the excitation in the outer layers is considerably lower and the value 10^{13} will be more correct. Such a great number of atoms in the state $2P$ leads to an optical thickness of the order of unity in the first two lines of the Balmer series. But the optical thickness in these lines will be sensitive to the variation of τ_0. If, for example, $\tau_0 = \frac{1}{2}$, the nebula will be transparent for each Balmer line.

* For the numerical evaluation of $\frac{1}{2}(K_1 + K_2)$ it is necessary to know the value of $\omega = \kappa/k$. According to Sugiura, the value of k beyond the limit of Lyman series is $0 \cdot 54 \times 10^{-17}$ per atom. The value of κ is determined by the formula

$$\kappa = \frac{c^3}{8\pi^{3/2}\nu_a{}^3 v} \frac{g_n}{g_m},$$

where c is the velocity of light, g_n and g_m are the weights of the stationary states, and v is a function of temperature T:

$$v = \sqrt{\frac{2k_1 T}{m}},$$

where m is the mass of hydrogen and k_1 Boltzmann's constant. The above formula for κ holds when the Doppler effect is the main cause of broadening of the line. For $T = 10 \cdot 000°$ we find $\omega \simeq 10^4$.

It may be mentioned here that there is another cause that will increase the optical thickness of the nebula in the lines of the Balmer series. It is well known that atoms in nebulæ have a tendency to become accumulated in metastable states. The state $2S$ of hydrogen is the metastable one. And the atoms in this state are capable of absorbing the Balmer quanta. If the metastability is a strict one, *i.e.* if the transitions from the metastable state to the lowest states are quite impossible, then Boltzmann's Law

$$\frac{n_2}{n_1}=\frac{g_2}{g_1}e^{-\frac{h\nu_2}{kT}}, \tag{40}$$

as Rosseland has shown, will govern the ratio n_2/n_1. But owing to the existence of the small probability of the transition $2S \rightarrow 1S$, the formula for n_2/n_1 will have another form, which we consider in a further paper. It is only important to remark that our values for n_2 are only the lower limits.

In view of these facts, the Balmer series of hydrogen is in a situation quite different from that of other subordinate series. While in all lines of other series the nebula is perfectly transparent (for example in the lines of the Paschen series), in the first members of the Balmer series the optical thickness of the nebula approaches, if not reaches, unity.

Radiation-pressure in the Outer Parts of the Nebula.—The greater part of the ultra-violet radiation of the star is transformed by the nebula into L_α-quanta. The flux of radiation emitted by the nebula will consist chiefly of L_α-quanta. For sufficiently large τ_0 each ultra-violet quantum will give rise to an L_α-quantum, and the flow of L_α-radiation from the nebula will be of the order $\frac{\nu_\alpha}{\nu_c}\pi S$. On the inner surface of the nebula the flux of L_α-radiation will be practically equal to zero, and the flux of ultra-violet radiation will be πS. Now the radiation-pressure in a layer of gas is proportional to the absorption coefficient. On the inner surface of the nebula the flux of radiation consists of ultra-violet quanta, for which the absorption coefficient is small. Therefore the radiation-pressure will not be very large. In the outer parts of the nebula, on the contrary, the flux of radiation consists chiefly of L_α-quanta, and the absorption coefficient is 10^4 times larger than in the case of ultra-violet quanta, while the flux of radiation is of the same order. The radiation-pressure, or, more exactly, the gradient of radiation-pressure, will be here 10^4 times larger than on the inner boundary of the ring. It is physically clear that for τ_0 large the net flux of L_α-radiation πF_α in the outer layer of the ring will be determined by

$$\pi F_\alpha=\left(\frac{r_*}{r_n}\right)^2\frac{2\pi h\nu_\alpha}{c^2}\int_{\nu_0}^{\infty}\frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}}-1},$$

where r_* and r_n are respectively the radius of the central star and the radius of nebula. The average impulse received by a hydrogen atom in normal state per second from L_α-quanta will be

$$\frac{k\pi F_a}{c}=\left(\frac{r_*}{r_n}\right)^2\frac{2k\pi h\nu_a}{c^3}\int_{\nu_0}^{\infty}\frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}}-1}.$$

The impulse received by each hydrogen atom from the gravitational field of the central star per second is

$$g(r_*/r_n)^2 m,$$

where g is the gravitational acceleration on the surface of the central star. But not only the normal hydrogen atoms, but the protons also are subject to gravitational force. Therefore the ratio μ of repulsive force R to the attractive force G is given by

$$\mu=\frac{R}{G}=\frac{k\pi}{mg\left(1+\frac{n^+}{n_1}\right)}\frac{2h\nu_a}{c^3}\int_{\nu_0}^{\infty}\frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}}-1},$$

where $\frac{n^+}{n_1}$ is the ratio of the number of protons per cubic centimetre to the number of normal hydrogen atoms in the same volume. Even in the case when we put

$$n^+/n=500,$$

which value is probably too high,* we obtain for $T=40{,}000°$

$$\mu=10^{10}/g.$$

The value of g for the nuclei of planetary nebulæ will be much larger than that for the Sun. But it is very improbable that it may reach 10^{10} cm. sec^{-2}. Therefore we may conclude that when the optical thickness of the nebula in the ultra-violet region is not exceedingly small, the radiation-pressure will be the dominant factor in the exterior parts of the nebula.

Distribution of Light over the Disc.—It would be useless to compute the distribution of ultra-violet light on L_α-radiation over the disc of the nebula, for we have no experimental data on this subject. But it is possible to say something about the distribution of energy over the monochromatic images of the nebula in the higher members of the Balmer series (H_δ, H_ε, H_η, etc.). In fact, the absorption coefficient in these lines will be so small that the nebula will be transparent in these lines in spite of the large number of hydrogen atoms in the second level. If $B'(t')$ is the *Ergiebigkeit* of the nebula in the given Balmer line, and t' is the corresponding optical depth, the distribution of radiation in the higher members of the Balmer series will be governed by the law:

$$I(o,\theta)=C_1\int_0^{t_0'}B(t')\sec\theta dt',$$

where θ is the angular distance of the point under consideration from the

* Zanstra, *Zs. für Astrophysik*, 2, 337, 1931.