A PROBLEM IN THE THEORY OF EIGENVALUES

In a certain special case (oscillating string, the natural boundary conditions) the spectrum of eigenvalues determines uniquely the differential equation to which it corresponds (in Schrödinger's theory, the "equation of amplitudes").

In those fields of theoretical physics (wave mechanics, theory of oscillations) where eigenvalue problems arise, the question of the uniqueness of determination of the mechanical system (i.e., of the Hamiltonian) by the set of the eigenvalues of the corresponding linear equation can be important. If a spectrum really completely determines the differential equation, then in principle it would become possible to determine the structure of an atomic system from the frequencies it is radiating or absorbing. This would mean solving a problem which is inverse to the Schrödinger problem. However, an approach to the general problem leads to many difficulties. Therefore, below we consider only a special case.

We prove that among all equations

$$\mu \frac{d^2\varphi}{dx^2} - q(x)\,\varphi + \alpha\varphi = 0,$$

where α is a "parameter" of eigenvalues, μ is a constant, $q(x)$ is a continuous function, for "natural boundary conditions"

$$\varphi'(0) = \varphi'(\pi) = 0$$

only the equation of the oscillating string

$$\mu \frac{d^2\varphi}{dx^2} + \alpha\varphi = 0$$

has the eigenvalues

$$\alpha_n = \kappa n^2.$$

§**1.** We start with the differential equation

$$(py')' - qy - \lambda ry + \alpha y = 0, \tag{1}$$

where λry is a perturbation term, q, r, p, p' are continuous functions of x and $p > 0$.

For the boundary conditions $y'(0) = y'(\pi) = 0$ the differential equation (1) has a countable set of eigenvalues which we can arrange in increasing order:

$$\alpha_1, \alpha_2, \alpha_3, \dots. \tag{2}$$

These eigenvalues are functions of λ. In this section our purpose is to show that these functions have no singularities on the real axis. It is sufficient to demonstrate that $\alpha_i(\lambda)$ are regular analytical functions of λ in the vicinity of the point $\lambda = 0$. The proof is by observation that the latter statement applies as well to the equation

$$(py')' + (q - \lambda_0 r)y - (\lambda - \lambda_0)ry + \alpha y = 0 \tag{3}$$

in the vicinity of arbitrary real λ_0. But equations (1) and (3) are identical.

First, let us suppose that $\alpha = 0$ is not an eigenvalue of the equation

$$(py')' + qy - \alpha y = 0. \tag{1'}$$

In this case, the differential operator

$$L(y) = (py')' + qy$$

has a Green function $G(x, \xi)$.

Then the power series

$$S(x, \xi, \lambda) = G_1(x, \xi) + \lambda G_2(x, \xi) + \lambda^2 G_3(x, \xi) + \dots, \tag{4}$$

in which

$$G_n(x, \xi) =$$
$$\int \cdots \int G(x, t_1)\, r(t_1)\, G(t_1, t_2)\, r(t_2) \dots r(t_{n-1})\, G(t_{n-1}, \xi)\, dt_1\, dt_2 \dots dt_{n-1}$$

converges inside some disk $|\lambda| < \rho$, since $G(x,\xi)$ and $r(x)$ are bounded. We have

$$S(x,\xi,\lambda) = \frac{k(x,\xi,\lambda)}{r(\xi)},$$

where $k(x,\xi,\lambda)$ is the resolvent of the kernel $G(x,\xi)\,r(x)$. The function $S(x,\xi,\lambda)$ represents for $|\lambda| < \rho$ the Green function of the differential operator

$$L(y) - \lambda\, ry = (py')' - qy - \lambda\, ry.$$

The eigenvalues of equation (1) are the null-points of the Fredholm denominator of the kernel $S(x,\xi,\lambda)$. This means that for $|\lambda| < \rho$ we can determine these eigenvalues from the equation:

$$D(\alpha,\lambda) = 1 - \frac{1}{1!}D_1(\lambda)\,\alpha + \frac{1}{2!}D_2(\lambda)\,\alpha^2 - \frac{1}{3!}D_3(\lambda)\,\alpha^3 + \ldots = 0, \qquad (5)$$

where

$$D_n(\lambda) = \int\cdots\int \begin{vmatrix} S(x_1,x_1,\lambda) & S(x_1,x_2,\lambda) & \cdots & S(x_1,x_n,\lambda) \\ S(x_2,x_1,\lambda) & S(x_2,x_2,\lambda) & \cdots & S(x_2,x_n,\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ S(x_n,x_1,\lambda) & S(x_n,x_2,\lambda) & \cdots & S(x_n,x_n,\lambda) \end{vmatrix} dx_1\, dx_2 \ldots dx_n.$$

The series (5) is uniformly convergent for $|\lambda| \leq \rho - \varepsilon$, where ε is a positive number and for all finite values of α, see [1]. Consequently it is an analytic function of two variables, and the regions of convergence are the whole α-plane and the circle $|\lambda| < \rho$.

If we expand $D(\alpha,\lambda)$ by powers of $\alpha - \alpha_i(0)$ and λ and take into account that $\alpha_i(0)$ is a simple root of the equation $D(\alpha, 0) = 0$, we will find that the constant term of the expansion vanishes, but the coefficient of the term $[\alpha - \alpha_i(0)]$ is nonzero. According to the theorem about implicit functions, we can state that within some convergence circle the function $\alpha_i(\lambda)$, i.e., the root of the equation $D(\alpha,\lambda) = 0$ which coincides with $\alpha_i(0)$ at $\lambda = 0$, can be expanded into series by powers of λ. Thus the eigenvalues are analytic functions of the perturbation parameter λ, provided $\alpha = 0$

is not an eigenvalue of equation (1′). The last condition, however, is not essential. Indeed, assuming $\alpha = 0$ is an eigenvalue of (1′), let us denote by k the nearest eigenvalue by module. Then we consider the equation

$$(py')' - \left(q + \frac{k}{2}\right) y + \beta y = 0, \tag{6}$$

for which β is not an eigenvalue. The eigenvalues of the equation

$$(py')' - \left(q + \frac{k}{2}\right) y - \lambda r y + \beta y = 0 \tag{7}$$

are analytic functions of λ in the vicinity of $\lambda = 0$. However, the eigenvalues of (7) and (1) differ merely by a constant $\frac{k}{2}$. Therefore, they are also analytic functions of λ.

We have proved that for any value of λ the eigenvalues of equation (1) are analytic functions of λ.

§**2.** The same reasoning shows that $D(x, \xi, \alpha, \lambda)$ (Fredholm's numerator) is also an analytic function of α and λ in the whole α-plane and in some circle $|\lambda| < \rho$.

We denote the normalized eigenfunctions of equation (1′) by

$$\varphi_1(x, \lambda), \varphi_2(x, \lambda), \dots . \tag{8}$$

It is well known that the products $\varphi_i(x, \lambda)\, \varphi_i(\xi, \lambda)$ are residues of the resolvent

$$\Gamma(x, \xi; \alpha, \lambda) = \frac{D(x, \xi, \alpha, \lambda)}{D(\alpha, \lambda)}$$

at the point $\alpha = \alpha_i(\lambda)$. Thus we have

$$\varphi_i(x, \lambda)\, \varphi_i(\xi, \lambda) = \frac{1}{2\pi i} \int_C \Gamma(x, \xi; \alpha, \lambda)\, d\alpha, \tag{9}$$

where C is a curve on the α-plane which encircles the point $\alpha_i(\lambda)$, but no other $\alpha_j(\lambda)$ $(j \neq i)$.

When λ belongs to the region $|\lambda| < a < \rho$ where a is a positive number to be selected later, each eigenvalue remains in some region B_i. It is easy to see that for sufficiently small a no region B_j has points in common with

B_i $(j \neq i)$. This follows from the following two facts: 1) in the case of limited changes of λ all variations of the eigenvalues $\alpha_i(\lambda)$ are uniformly bounded, and 2) if $\alpha_1(\lambda), \ldots, \lambda_N(\lambda)$ are the first N eigenvalues that are analytic functions of λ, we can take N so large that $\alpha_{N+1}(\lambda), \ldots$ for every $|\lambda| < \rho$ remain greater than $\alpha_i(\lambda)$ for the same λ.

Since $\alpha_i(0)$ are all distinct, we can take a so small that for $|\lambda| < a$ all $\alpha_i(\lambda)$ $(i = 1, \ldots, N)$ are regular and the regions B_i are pairwise disjoint. Now we can choose C in such a way that it encircles B_i, but does not encircle any point of B_j $(j \neq i)$ (see Fig. 1). Formula (9) then shows that for sufficiently small λ, the function $\varphi_i(x, \lambda)\, \varphi_i(\xi, \lambda)$ depends analytically on λ. From this we can conclude that $\varphi_i(x, \lambda)$ is also an analytic function of λ.

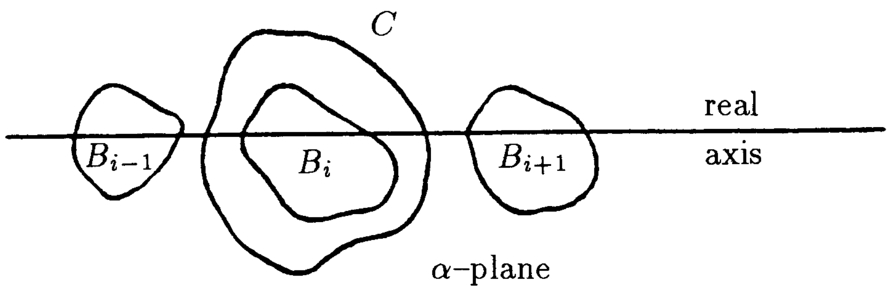

Fig.1.

For further reasoning the expressions of the perturbed eigenvalues are essential. We write here only the first three terms:

$$\alpha_i(\lambda) = \alpha_i(\lambda_0) + (\lambda - \lambda_0)\, \varepsilon_{ii}(\lambda_0) + (\lambda - \lambda_0)^2 \sum_{j=1}^{\infty}{}' \frac{\varepsilon_{ij}(\lambda_0)}{\alpha_i(\lambda_0) - \alpha_j(\lambda_0)} + \cdots, \quad (10)$$

where $\sum'$ denotes summation with the case $i = j$ excluded, and

$$\varepsilon_{ij}(\lambda_0) = \int_0^{\pi} r(x) \varphi_i(x, \lambda_0) \varphi_j(x, \lambda_0)\, dx. \quad (11)$$

§**3.** Let us now assume for a moment that the equation

$$\mu \frac{d^2\varphi}{dx^2} - r(x)\, \varphi + \alpha\varphi = 0 \quad (12)$$

has the same system of eigenvalues as the equation

$$\kappa \frac{d^2\varphi}{dx^2} + \alpha\varphi = 0 \tag{12'}$$

when the boundary conditions are $\varphi'(0) = \varphi'(\pi) = 0$.

Then of necessity, $\mu = \kappa$. This follows from the comparison of the asymptotic expressions for the eigenvalues

$$\alpha_n = n^2\mu + O(1) \quad \text{of (12) and} \quad \alpha_n = n^2\kappa + O(1) \quad \text{for (12')}.$$

Let us write the equations (12), (12′) for the case $\lambda = 0$ and $\lambda = 1$

$$\kappa \frac{d^2\varphi_i(x,0)}{dx^2} + \alpha\varphi_i(x,0) = 0,$$

$$\kappa \frac{d^2\varphi_i(x,1)}{dx^2} - r(x)\,\varphi_i(x,1) + \alpha_i\varphi_i(x,1) = 0.$$

We multiply the first of these equation by $\varphi_i(x,1)$, the second by $\varphi_i(x,0)$, subtract and integrate the result. Then according to the Green formula we obtain

$$\int r(x)\,\varphi_i(x,0)\,\varphi_i(x,1)\,dx = 0. \tag{13}$$

For large values of i we have asymptotic formulas:

$$\varphi_i(x,0) = \sqrt{\frac{2}{\pi}}\cos ix + O\left(\frac{1}{i}\right),$$

$$\varphi_i(x,1) = \sqrt{\frac{2}{\pi}}\cos ix + O\left(\frac{1}{i}\right),$$

yielding the asymptotic expression

$$\varphi_i(x,0)\,\varphi_i(x,1) = \frac{2}{\pi}\cos^2 ix + O\left(\frac{1}{i}\right) = \frac{1}{\pi}[1 + \cos 2ix] + O\left(\frac{1}{i}\right).$$

Now from

$$\lim_{i\to\infty}\int_0^\pi r(x)\,\cos 2ix\,dx = 0 \quad \text{and} \quad \lim_{i\to\infty}\int_0^\pi r(x)\,O\left(\frac{1}{i}\right)\,dx = 0$$

and from (13) we conclude that

$$\lim_{i\to\infty} \frac{1}{\pi} \int r(x)\,\varphi_i(x,0)\,\varphi_i(x,1)\,dx = \frac{1}{\pi} \int r(x)\,dx = 0.$$

However, since $\varphi_1(x,0) = 1\,/\sqrt{\pi}$, we can write the expansion of $\alpha_1(\lambda)$ for $\lambda_0 = 0$ according to (10) in the form

$$\alpha_1(\lambda) = -\lambda^2 \sum_{i=2}^{\infty} \frac{\varepsilon_{1j}^2}{\alpha_j(0)} + \dots . \tag{14}$$

From this we conclude that for sufficiently small λ, the value of $\alpha_1(\lambda)$ is negative.

Differentiating (14) we see that the derivative of $\alpha_1(\lambda)$ for sufficiently small positive values of λ is negative. But we have already adopted that $\alpha_1(0) = \alpha_1(1) = 0$. Therefore, $\alpha_1'(\lambda)$ somewhere in the interval $(0,1)$ is positive and changes its sign.

Let $\lambda = \delta$ be the point where $\alpha'(\delta) = 0$. Since $\alpha_1'(0) = 0$ we find that at some point δ_1 the second derivative must vanish.

According to (10), this means

$$\sum_{j=2}^{\infty} \frac{\varepsilon_{1j}^2(\delta_1)}{\alpha_1(\delta_1) - \alpha_j(\delta_1)} = 0.$$

Since all terms here are negative, we obtain

$$\varepsilon_{1j}^2(\delta_1) = 0, \quad \varepsilon_{1j}(\delta_1) = 0 \quad (j \neq 1). \tag{15}$$

However, according to (11) ε_{1j} are the coefficients of expansion of the function $q(x)\,\varphi(x_i,\delta_1)$ by the series of orthogonal functions $\varphi_j(x,\delta_1)$ $(j = 1, 2, \dots)$. The system is complete and, therefore, it follows from (15) that

$$r(x)\,\varphi_1(x,\delta_1) = C\varphi_1(x,\delta_1)$$

or $r(x) = C$.

On the other hand $\int_0^{\pi} r\,dx = 0$ implying $C = 0$. It follows that $r(x) = 0$.

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REFERENCES

1. R. Courant und D. Hilbert, *Methoden der Mathematischen Physik*, 126, 1924.

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